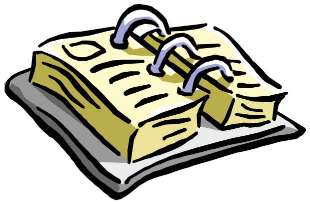


New Approaches to Phasing of Resources

Will Jarvis
OSD PA&E

Presented at the 34th Annual DoDCAS, Williamsburg, VA, February 2, 2001

Introduction

- Rate of progress on a project is affected by dependencies among the problems being solved. For example,
    - Some problems cannot be worked until others are finished.
    - Some problems can be worked simultaneously.
    - New problems may surface as old ones are being worked.

Introduction

- If a project ramps up too quickly to maximum loading, then resources are wasted before work is really uncovered and available for doing.
- If a project ramps up to slowly then budget overruns occur late in the project as timelines are compressed.
- Is there an optimum way to allocate resources to a project?

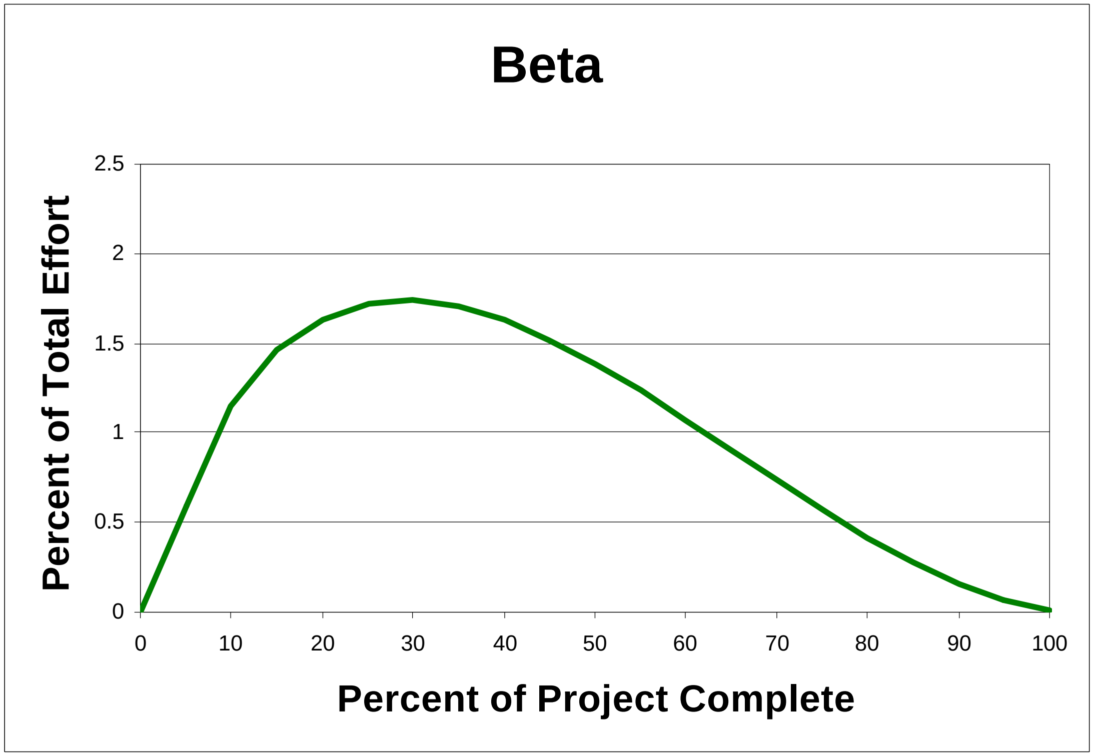
Introduction

- We examine four methods of resource allocation:
  - Beta Curve
  - Rayleigh Model
  - $Sech^2$ Model
  - Damped Sine Model

Beta

- The beta curve is often used to empirically fit manpower patterns,

$$\frac{dW(t)}{dt} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} t^{a-1} \bullet (1-t)^{b-1} \qquad 0 < t < 1$$



Beta

- The beta curve provides great flexibility; however, a theoretical justification for use of the curve is lacking.
- Norden introduced the Rayleigh curve, asserting that a project is a set unsolved problems.
- Later Parr, introduced the notion of *visible* unsolved problem space.

Rayleigh Model

- A project involves solving some fixed number of problems.
- The rate at which problems are solved is jointly proportional to (1) the level of skill available, *p(t)*; and (2) the fraction of problems left to solve.

$$\frac{dW(t)}{dt} = p(t) \bullet (1 - W(t))$$

- On integration,

$$W(t) = 1 - \exp\left(-\int^{t} p(t)\,dt \right) \qquad \textit{Eqn. R1}$$

Rayleigh Model

- Assume a linear learning rate.

$$p(t) = \acute{a} \cdot t$$ *Eqn. R2*

- The Rayleigh curve follows (*substitute Eqn R2 into Eqn R1, then differentiate*).

$$\frac{dW(t)}{dt} = \acute{a} \cdot t \cdot \exp\left[-\frac{\acute{a} \cdot t^2}{2}\right]$$ *Rayleigh Curve*

Rayleigh Model

- “Spreading” an Estimate
  - Time of Peak Staffing is close to development time, so let

$$\left(\frac{d^2W(t)}{dt^2}\right)_{t=t_d} = -\alpha^2 t_d{}^2 e^{-\frac{\alpha t_d{}^2}{2}} + \alpha e^{-\frac{\alpha t_d{}^2}{2}} = 0$$

  - Solving for α, we get,

$$\alpha = \frac{1}{t_d{}^2}$$

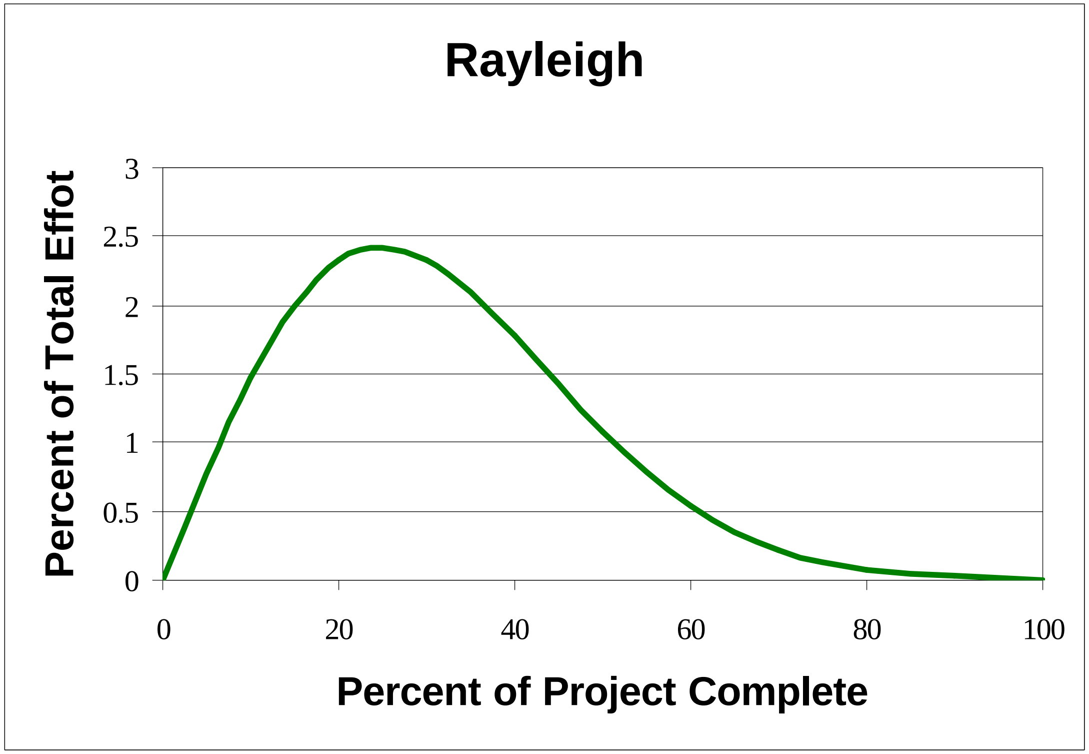
  - For the Rayleigh distribution, the peak occurs at the 39th percentile so,

$$K = \frac{E}{0.39}$$

Rayleigh Model

- In terms of development time (t_d) and total effort (E), resource consumption is expressed as,

$$\frac{dW(t)}{dt} = \frac{E}{0.39} \bullet \left(\frac{1}{t_d^{\ 2}} \cdot t \cdot \exp\left[-\frac{t^2}{2 \cdot t_d^{\ 2}} \right] \right)$$



Rayleigh Model

- Main assumptions of Putnam-Norden Rayleigh Model:
  - Initial <u>ramp-up</u> is due to a linear learning curve.
  - Exponential <u>tail-off</u> is due to exhaustion of the problem space -- that is, the rate at which work is done is proportional to the amount of work left.
- Parr criticizes Putnam and Norden’s first assumption, arguing that it “confuses intrinsic constraints” on the rate at which work can be accomplished with “management’s economically governed choices on how to respond to these constraints”.

Parr Sech^2 Model

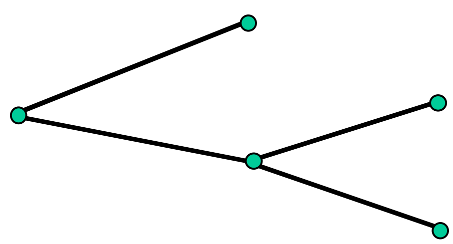
- Let *W(t) = proportion of problems which have been solved*, as with the Putnam-Norden Model.
- Introduce *V(t) = proportion of* <u>*visible*</u> *unsolved problems.*
- Parr asserts that the rate of progress on a project is governed by dependencies among the problems that must be solved.
  - Some problems must be solved in sequential order.
  - Other problems can be solved in parallel.

Parr Sech^2 Model

- If we consider a short time interval that encloses the solution of just one problem, then it is immediate that

$$W\left(t^{+}\right) = W\left(t^{-}\right) + 1$$

- Now, Parr assumes that the dependency relation among problems is a binary tree. That is, the solution of a problem leads to no further problems, or else leads to two new problems.



Parr Sech^2 Model

- In equation form,

$$V(t^+) = V(t^-) + 1 \qquad \textit{if} \text{ problem has dependents}$$
$$V(t^+) = V(t^-) - 1 \qquad \textit{if} \text{ problem has no dependents}$$

- Second assumption:
  - "... the probability that the most recently solved problem has no dependents is linear in the number of problems solved."

- In equation form, this becomes

$$V(t^+) = V(t^-) + 1 \qquad \text{with probability } 1 - {W(t^-)}/{N}$$
$$V(t^+) = V(t^-) - 1 \qquad \text{with probability } {W(t^-)}/{N}$$

N is the total number of problems to be solved.

Parr Sech^2 Model

- Taking expectations yields,

$$E\left[V\left(t^{+}\right)\right] = V\left(t^{-}\right) + 1 - 2\frac{W\left(t^{-}\right)}{N}$$

- In words: *The expected change in the number of visible problems decreases linearly with the work completed.*

- Each time a problem is solved, we expect the size of the visible problem space to grow as

$$\Delta V(t) = 1 - 2W(t)$$

N disappears through normalization.

Parr Sech^2 Model

- Now, given that the rate at which problems are solved is *dW(t)/dt*, then

$$\frac{dV(t)}{dt} = \frac{dW(t)}{dt} \cdot (1 - 2W(t)) \qquad \textit{Eqn. S1}$$

- In words: *The rate of change in the number of visible unsolved problems is jointly proportional to (1) the rate of solving problems; and (2) the expected change in the number of visible unsolved problems per problem solved.*

Parr Sech^2 Model

- Parr’s second assertion is that “the rate at which work can be usefully input ... is proportional to *V(t)*.”

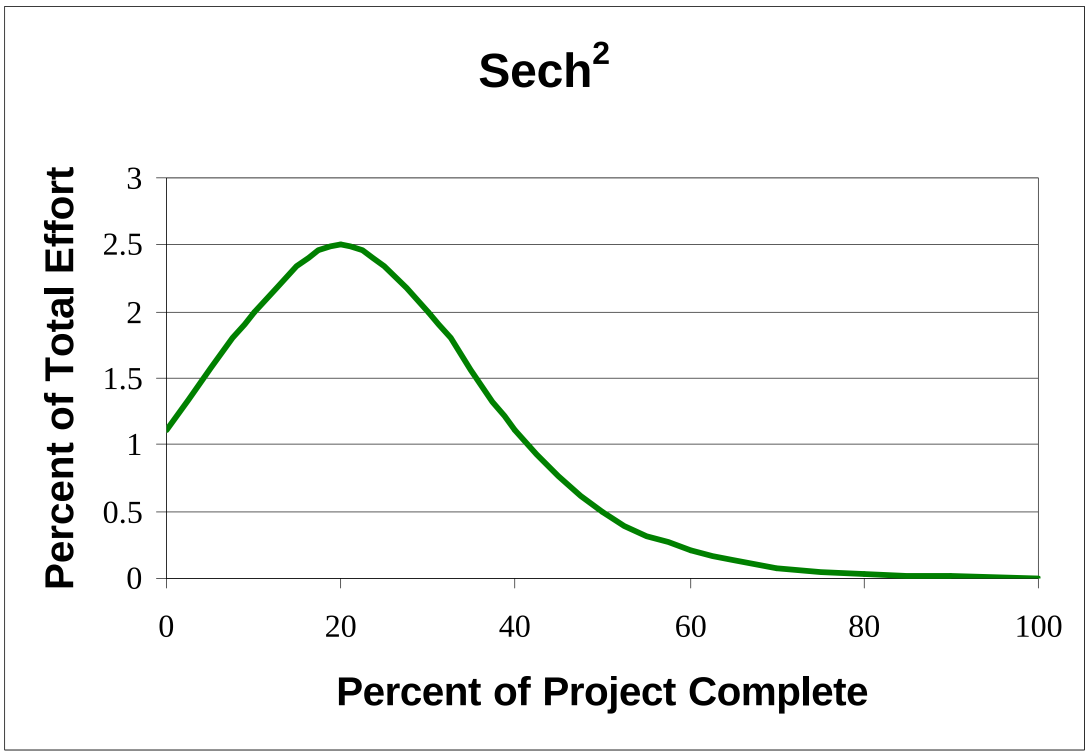
- In equation form,

$$\frac{dW(t)}{dt} = \acute{a} \cdot V(t) \qquad \textit{Eqn. S2}$$

Parr Sech^2 Model

- Solution, found by solving Eqns S1 and S2, is

$$V(t) = \frac{A\exp(-\acute{a}t)}{(1 + A\exp(-\acute{a}t))^2} = \frac{1}{4} \cdot \operatorname{sech}^2\left(\frac{(\acute{a}t + c)}{2}\right)$$



Damped Sine Model

- As before,
  - Let *W(t) = proportion of problems which have been solved*, (same as Putnam-Norden Model).
  - Let *V(t) = proportion of visible unsolved problems.*
- Two assumptions:
  - Rate at which problems are uncovered is proportional to the difference between (a) the amount of work remaining, and (b) the amount of visible unsolved problems.
  - Rate at which problems are solved is proportional to the number of visible unsolved problems.

Damped Sine Model

- In equation form, our assumptions can be restated as

$$\dot{V} = (1 - W) - V$$
$$V = \dot{W}$$

For clarity, the constants of proportionality are omitted.

- Differentiating the first equation further, we get

$$\ddot{V} = -\dot{W} - \dot{V}$$
$$V = \dot{W}$$

- Together, these two equations yield

$$\ddot{V} + \dot{V} + V = 0$$

Damped Sine Model

- The general solution is of the form,

$$V = e^{-\alpha t}(C_1 \cos bt + C_2 \sin bt)$$

Constants α and b are functions of proportionality constants in original set of differential equations.

- With the constraint that V(0) = 0, then the general solution reduces to

$$V = e^{-\alpha t} \sin bt$$

Later, we normalize the solution. So for now, we can set $C_2 = 1$, without loss in generality.

- As with the Rayleigh, we associate the time of peak staffing, with the end of the development phase (t_{dev}). So,

$$\dot{V}(t_{dev}) = -\alpha e^{-\alpha t_{dev}} \sin bt_{dev} + be^{-\alpha t_{dev}} \cos bt_{dev} = 0$$

Damped Sine Model

- Solution of this equation yields

$$\acute{a} = b\,\frac{\cos bt_{dev}}{\sin bt_{dev}}$$

- Now, we introduce a project endpoint as t_{end}. We require that

$$V(t_{end}) = e^{-\acute{a}t_{end}} \sin bt_{end} = 0$$

or

$$\sin bt_{end} = 0$$

or

$$b = n\,\frac{p}{t_{end}}$$

For our purpose, *n = 1.*

Damped Sine Model

- The normalization factor is derived as,

$$C = \int_{0}^{t_{end}} V \cdot dt$$

$$= \int_{0}^{t_{end}} e^{-\acute{a}t} \sin bt \cdot dt$$

$$= \frac{e^{-\acute{a}t_{end}} (\acute{a} \sin bt_{end} - b \cos bt_{end}) + b}{\acute{a}^{2} + b^{2}}$$

Damped Sine Model

- The solution is now expressed in terms of project effort (*E* manmonths), development time (t_{dev}), project end-time (t_{end}):

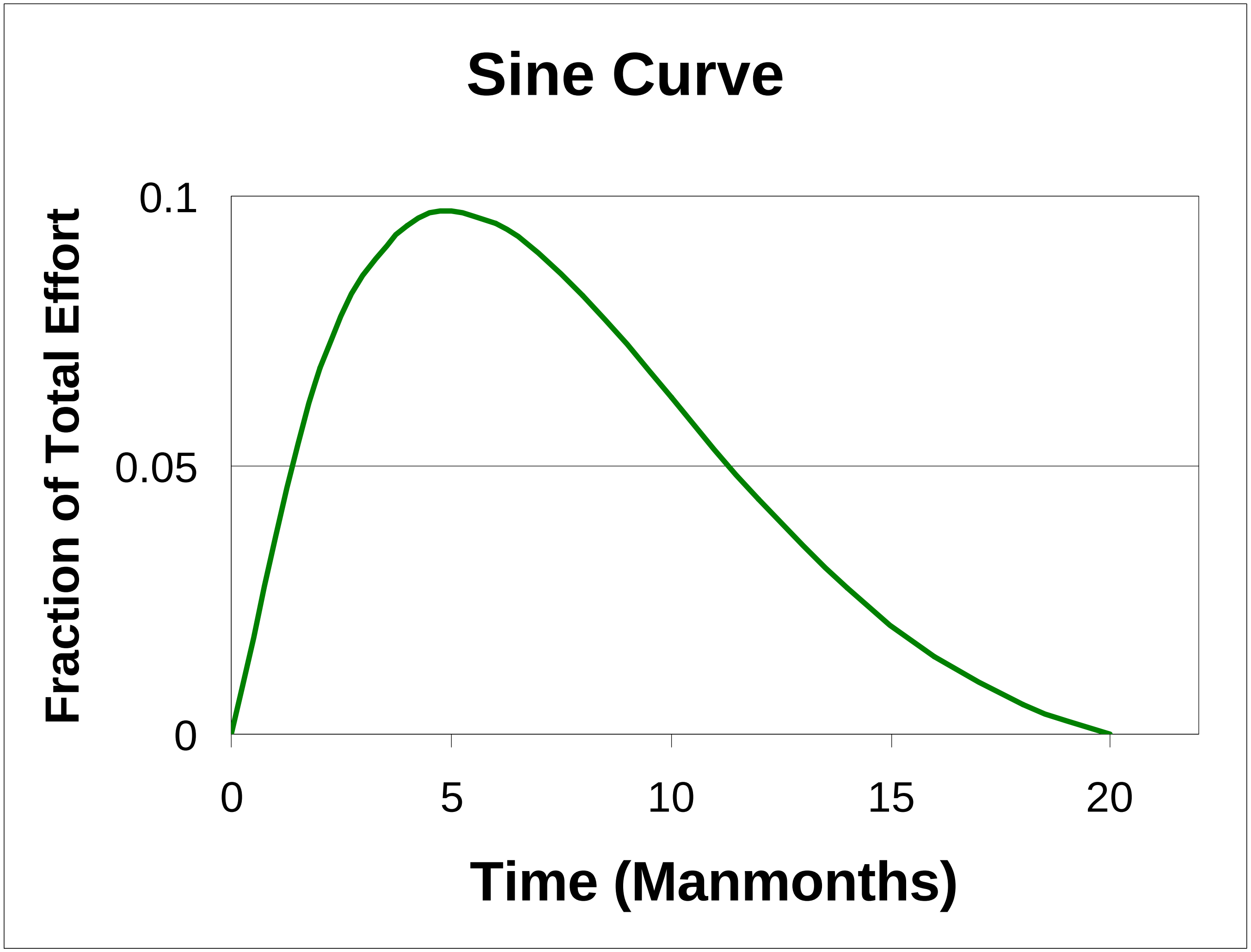
$$V = \frac{E}{C} e^{-\acute{a}t} \sin bt$$

where,

$$b = \frac{p}{t_{end}}$$

$$\acute{a} = b \frac{\cos bt_{dev}}{\sin bt_{dev}}$$

Damped Sine Model



Conclusions

- The Beta curve is a useful tool for fitting manpower patterns. However, it offers little understanding of how a problem-solving process behaves.
- The Rayleigh curve presumes a linear pattern of manpower buildup. It offers some understanding of how the process behaves once peak staffing occurs.
- For the Rayleigh, $Sech^2$, and Sine models, the decay in work rate is due to exhaustion of the problem space.

Conclusions

- In the $Sech^2$ and Sine models, resource allocation is fully unconstrained in the sense that program managers apply large amounts of input resources whenever there exists the potential for solving problems in parallel. In the Rayleigh model, resource allocation is linearly constrained during the buildup.

- Both the Rayleigh and $Sech^2$ curves have infinite tails, making them awkward to use. The Sine curve does not have infinite tails.

- The Rayleigh, $Sech^2$, and Sine curves offer insight into when we should expect peak resource expenditure to occur for any given project.

Next Steps

- Normalize all models, compare side-by-side.
- Evaluate alternative initial conditions.
    - Does changing the proportion of problems that are visible at the start of the project affect when peak staffing occurs?